

# Financial Markets with Volatility Uncertainty

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December 15, 2010

## Abstract

We investigate financial markets under model risk caused by uncertain volatilities. For this purpose we consider a financial market that features volatility uncertainty. To have a mathematical consistent framework we use the notion of *G-expectation* and its corresponding *G-Brownian motion* recently introduced by Peng (2007). Our financial market consists of a riskless asset and a risky stock with price process modeled by a geometric G-Brownian motion. We adapt the notion of arbitrage to this more complex situation and consider stock price dynamics which exclude arbitrage opportunities. Due to volatility uncertainty the market is not complete any more. We establish the interval of no-arbitrage prices for general European contingent claims and deduce explicit results in a Markovian setting.

*Key words and phrases:* Pricing of contingent claims, incomplete markets, volatility uncertainty, G-Brownian motion stochastic calculus

*MSC 2000 numbers:* 91G20

*JEL classification codes:* G13, D81, C61

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\*I thank Frank Riedel for valuable advice and Yongsheng Song for fruitful discussions. Financial support through the German Research Foundation (DFG) and the International Graduate College “Stochastics and Real World Models” Beijing–Bielefeld is gratefully acknowledged.

# 1 Introduction

Many choice situations exhibit ambiguity. At least since the Ellsberg Paradox the occurrence of ambiguity aversion and its effect on making economic decisions are well established. One possible way to model decisions under ambiguity is to use multiple priors. Instead of analyzing a problem in a single prior model as in the classical subjective expected utility approach one focuses on a multiple prior model to describe the agent's uncertainty about the right probability distribution.

These models have gained much attention in recent studies. The decision theoretical setting of multiple priors was introduced by Gilboa and Schmeidler (1989) and extended to a dynamic model by Epstein and Schneider (2003). Maccheroni, Marinacci, and Rustichini (2006) generalize this model to so-called variational preferences. Multiple priors appear naturally in monetary risk measures as introduced by Artzner, Delbaen, Eber, and Heath (1999) and its dynamic extensions, see Föllmer and Schied (2004) for an overview.

Most literature essentially concentrates on the modeling of multiple priors with respect to some reference measure.<sup>1</sup> The standing assumption is that all priors are at least (locally) absolutely continuous with regard to a given reference measure. This is often a technical assumption in order to simplify mathematics. However, it significantly affects the informative value of the multiple prior model.<sup>2</sup> In diffusion models, by Girsanov's theorem these multiple prior models only lead to uncertainty in the mean of the considered stochastic process, see Chen and Epstein (2002) or Cheng and Riedel (2010) for instance. Thus in Finance, these multiple prior models just lead to drift uncertainty for the stock price.

Obviously, one may imagine another source of uncertainty that involves the risk described by the standard deviation of a random variable. Especially in Finance this is of great relevance. For instance, the price of an option

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<sup>1</sup>See Nishimura and Ozaki (2007) or Riedel (2009), for example.

<sup>2</sup>The reference measure plays the role of fixing the sets of measure zero. This means that the decision maker has perfect knowledge about sure events which is obviously not always a reasonable property from an economic point of view. In particular, with a filtration satisfying the "usual conditions" which is mostly arranged one excludes economic interesting models since the decision maker consequently knows already at time zero what can happen and what not. This of course does not reflect reality well as for instance the recent incidents about the Greek government bonds illustrate.

written on a risky stock heavily depends on the underlying volatility. Also the value of a portfolio consisting of risky positions is strongly connected with the volatility levels of the corresponding assets. One major problem in practice is to forecast the prospective volatility process in the market. In this sense it appears quite natural to permit volatility uncertainty.<sup>3</sup> In the sense of risk measuring it is desirable to seize this risk.

Our paper may form the basis for working with volatility uncertainty in Finance. There are many problems like hedging of contingent claims under constraints, cf. Karatzas and Kou (1996), or portfolio optimization, cf. Merton (1990), that should also be formulated and treated in the presence of model uncertainty. For this one will also need an economic reasonable notion of arbitrage as developed in this paper.

Fernholz and Karatzas (2010) consider the question of outperforming the market when the assumption of no-arbitrage is not imposed and incorporate volatility uncertainty. Our purpose is to model volatility uncertainty on financial markets under the assumption of no-arbitrage. We set a framework for modeling this particular uncertainty and treat the pricing and hedging of European contingent claims. The setting is closely related to model risk which clearly matters in the sense of risk management. As we shall see allowing for uncertain volatilities will lead to incomplete markets, therefore effects the pricing and hedging of claims, and involves model risk. Our solution approach also provides a method to measure this risk. We consider European claims written on a risky stock  $S$  which additionally features volatility uncertainty. Roughly,  $S$  is modeled by the family of processes

$$dS_t^\sigma = rS_t^\sigma dt + \sigma_t S_t^\sigma dB_t$$

where  $B = (B_t)$  is a classical Brownian motion and  $\sigma_t$  attains various values in  $[\underline{\sigma}, \bar{\sigma}]$  for all  $t$ . In this setting we aim to solve

$$\sup_{P \in \mathcal{P}} E^P(H_T \gamma_T^{-1}) \quad \text{and} \quad \sup_{P \in \mathcal{P}} E^P(-H_T \gamma_T^{-1}) \quad (1)$$

where  $H_T$  denotes the payoff of a contingent claim at maturity  $T$ ,  $\gamma_T^{-1}$  a discounting, and  $\mathcal{P}$  presents a set of various probability measures describing the model uncertainty.

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<sup>3</sup>Volatility is very sensitive with respect to changing market data which makes its predictability difficult. It also reflects the market's sentiment. Currently high implicit volatility levels suggest nervous markets whereas low levels rather feature bullish mood. By taking many models into account one may protect oneself against surprising events due to misspecification.

It is by no means clear whether the expressions above are well-posed and how to choose  $\mathcal{P}$  in this case. As seen in Denis and Martini (2006) modeling uncertain volatilities leads to a set of priors  $\mathcal{P}$  which consists of mutually singular probability measures.<sup>4</sup> So when dealing with model uncertainty we need a consistent mathematical framework enabling us to work with processes under various measures at the same time. We utilize the framework of sublinear expectation and G-normal distribution introduced by Peng (2007) in order to model and control model risk.

We consider a Black–Scholes like market with uncertain volatilities, i.e., the stock price  $S$  is modeled as a geometric G–Brownian motion

$$dS_t = rS_t dt + S_t dB_t, \quad S_0 = x_0, \quad (2)$$

where the canonical process  $B = (B_t)$  is a G–Brownian motion with respect to a sublinear expectation  $E_G$ .  $E_G$  is called G–expectation. It also represents a particular coherent risk measure that enables to quantify the model risk induced by volatility uncertainty. For the construction see Peng (2007) or Peng (2010).

G–Brownian motion forms a very rich and interesting new structure which generalizes the classical diffusion model. It replaces classical Brownian motion to account for model risk in the volatility component. Each  $B_t$  is G-normal distributed which resembles the classical normal distribution. The function G characterizes the degree of uncertainty.

Throughout the paper we consider the case where G denotes the function  $y \mapsto G(y) = \frac{1}{2}\bar{\sigma}^2 y^+ - \frac{1}{2}\underline{\sigma}^2 y^-$  with volatility bounds  $0 < \underline{\sigma} < \bar{\sigma}$ . Each  $B_t$  has mean zero but an uncertain variance varying between the bounds  $\underline{\sigma}^2 t$  and  $\bar{\sigma}^2 t$ . So when  $B_t$  is evaluated by  $E_G$  we have  $E_G(B_t) = E_G(-B_t) = 0$  and  $E_G(B_t^2) = \bar{\sigma}^2 t \neq -E_G(-B_t^2) = \underline{\sigma}^2 t$ . Consequently, the stock’s volatility is uncertain and incorporated in the process  $(B_t)$ . The quadratic variation process is no longer deterministic. All uncertainty of  $B$  is concentrated in  $\langle B \rangle$ . It is absolutely continuous w.r.t. Lebesgue measure and its density satisfies  $\underline{\sigma}^2 \leq \frac{d\langle B \rangle_t}{dt} \leq \bar{\sigma}^2$ .

The related stochastic calculus, especially Itô’s integral, can also be established with respect to G–Brownian motion, cf. Peng (2010). Notions like martingales are replaced by G–martingales with the same meaning as one would expect from classical probability theory, cf. Definition A.16.

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<sup>4</sup>To understand how things are involved see Example 2.1 in Section 2.

Even though from the first view of Equation (2) it is hidden we are also in a multiple prior setting. Denis, Hu, and Peng (2010) showed that the G-framework developed in Peng (2007) corresponds to the framework of quasi-sure analysis.<sup>5</sup> They established that the sublinear expectation  $E_G$  can be represented as an upper expectation of classical expectations, i.e., there exists a set of probability measures  $\mathcal{P}$  such that  $E_G[X] = \sup_{P \in \mathcal{P}} E^P[X]$ .

It should be mentioned that the stipulated dynamics for the stock price in (2) imply that the discounted stock price is a symmetric G-martingale. The word “symmetric” implies that the corresponding negative process is also a G-martingale which is not necessarily the case in the G-framework.<sup>6</sup> For this stock price model we prove that the induced financial market does not admit any arbitrage opportunity.<sup>7</sup> In addition, this accords with classical Finance in which problems like pricing and hedging of claims are solved with respect to a risk neutral martingale measure such that the discounted price process becomes a (local) martingale.

The notion of G-martingale plays an important role in our analysis (see page 36 in the appendix for a deeper understanding).

By Soner, Touzi, and Zhang (2010a) a G-martingale  $(M_t)$  solves the following dynamic programming principle, see also Appendix A.3:

$$M_t = \operatorname{ess\,sup}_{Q' \in \mathcal{P}(t, Q)} E^{Q'}(M_s | \mathcal{F}_t) \quad Q - a.s., \quad t \leq s,$$

where  $\mathcal{P}(t, Q) := \{Q' \in \mathcal{P} | Q' = Q \text{ on } \mathcal{F}_t\}$ . Thus, a G-martingale is a multiple prior martingale as considered in Riedel (2009). The dynamic programming principle states that a G-martingale is a supermartingale for all single priors and a martingale for an optimal prior. Using the G-martingale representation theorem of Song (2010b), cf. Theorem A.19, and Remark A.18 we obtain that a symmetric G-martingale will be a martingale with respect to all single priors involved. So, also from this point of view the imposed dynamics on  $S$  in (2) are economic reasonable.

In such an ambiguous financial market we analyze European contingent claims concerning pricing and hedging. We extend the asset pricing to markets with volatility uncertainty. The concept of no-arbitrage will play a

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<sup>5</sup>See Denis and Martini (2006), for example.

<sup>6</sup>This characteristic forms a fundamental difference to classical probability theory. Its effect is also reflected in the results of Section 3.

<sup>7</sup>In an accompanying paper we will give further economic verification for the dynamics in (2).

major role in our analysis. Due to the additional source of risk induced by volatility uncertainty the classical definition of arbitrage is no longer adequate. We introduce a new arbitrage definition that fits to our multiple prior model with mutually singular priors. We verify that our financial market does not admit any arbitrage opportunity in this modified sense.

Using the concept of no-arbitrage we establish detailed results providing a better economic understanding of financial markets under volatility uncertainty. We determine an interval of no-arbitrage prices for general contingent claims. The bounds of this interval – the upper and lower arbitrage prices  $h_{up}$  and  $h_{low}$  – are obtained as the expected value of the claim's discounted payoff with respect to  $G$ -expectation, see (1). They specify the lowest initial capital required to hedge a short position in the claim, or long position, respectively.<sup>8</sup>

Since  $E_G$  is a sublinear expectation we know that  $h_{low} \neq h_{up}$  in general which verifies the market's incompleteness. All in all, any price being within the interval  $(h_{low}, h_{up})$  is a reasonable initial price for a European contingent claim in the sense that it does not admit arbitrage.

In a Markovian setting when the claim's payoff only depends on the current stock price of its underlying we deduce more structure about the upper and lower arbitrage prices via the so-called Black–Scholes–Barenblatt PDE. We derive an explicit representation for the corresponding superhedging strategies and consumption plans.<sup>9</sup> In the special situation when the payoff function exhibits convexity (concavity) the upper arbitrage price solves the classical Black–Scholes PDE with volatility equal to  $\bar{\sigma}$  ( $\underline{\sigma}$ ), and, vice versa concerning the lower arbitrage price. This corresponds to a worst-case volatility analysis as in El Karoui and Quenez (1998) and Avellaneda, Levy, and Paras (1995).

In the Markovian setting the same results were also established in Avellaneda, Levy, and Paras (1995). However, the mathematical framework in Avellaneda, Levy, and Paras (1995) is rather intuitive and presumably not sufficient for a general study. Our analysis thus provides a rigorous foundation for the results in Avellaneda, Levy, and Paras (1995). A recent preprint of Soner, Touzi, and Zhang (2010b) also contains relevant results concerning

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<sup>8</sup>The expression on the left hand side in (1) may be interpreted as the ask price the seller is willing to accept for selling the claim, whereas the other represents the bid price the buyer is willing to pay.

<sup>9</sup>It is also called side-payments, cf. Föllmer and Schied (2004).

uncertain volatility.<sup>10</sup> Denis and Martini (2006) also obtained the identity  $h_{up} = E_G(H_T \gamma_T^{-1})$  for the case when  $H_T \in \mathcal{C}_b(\Omega)$  by using quasi-sure analysis.<sup>11</sup> We formulate the financial market in the presence of volatility uncertainty using the G-framework of Peng (2007). Utilizing the set of multiple priors  $\mathcal{P}$  induced by  $E_G$  we are able to investigate the market by a more powerful approach – a no-arbitrage concept which also accounts for volatility uncertainty. Only due to the properties of G-Brownian motion we are also able to obtain explicit results like the PDE derivation in the Markovian setting. In particular we realize the PDE of Black and Scholes (1973) as a special case when the volatility uncertainty is set equal to zero. Furthermore, we consider the lower arbitrage price  $h_{low}$  and obtain the identities for claims with  $H_T \in L_G^p(\Omega_T), p \geq 2$ .<sup>12</sup> Both bounds together form the basis for an economic reasonable price for the claim.

The paper is organized as follows. Section 2 introduces the financial market we focus on and extends terminologies from mathematical finance. Section 3 stresses on the concept of no-arbitrage and the pricing of contingent claims in the general case. In Section 4 we restrict ourselves to the Markovian setting and obtain similar results as in Avellaneda, Levy, and Paras (1995). Section 5 concludes. Appendix A introduces to the notions of sublinear expectation. It summarizes the necessary definitions, constructions and associated results from the original sources as far as we used it in the introduction and preceding sections. So the appendix serves as a reference work for the reader to obtain a deeper understanding of the mathematics. Whenever we use a new item from the G-framework in the previous sections we will give a brief and rough explanation in order to facilitate a first reading without knowing the framework being presented in the Appendix A.

## 2 The market model

We aim to analyze financial markets that feature volatility uncertainty. The following example (see also Soner, Touzi, and Zhang (2010b)) illustrates some issues that arise when we deal with uncertain volatilities.

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<sup>10</sup>The authors use an approach different from ours.

<sup>11</sup> $\mathcal{C}_b(\Omega)$  denotes the space of bounded continuous functions on the path space.

<sup>12</sup> $L_G^p(\Omega_T)$  represents a specific space of random variables for which the G-expectation can be defined.  $\mathcal{C}_b(\Omega)$  is contained in  $L_G^p(\Omega_T)$ , see Peng (2010) or Equation (9) in Appendix A.1.



**Example 2.1** Let  $(B_t)$  be a Brownian motion with regard to some measure  $P$  and consider the price process modeled as  $dS_t^\sigma = \sigma_t dB_t$ ,  $S_0 = x$ , for various processes  $\sigma$ . If  $\hat{\sigma} \neq \tilde{\sigma}$  we have  $\langle S^{\hat{\sigma}} \rangle \neq \langle S^{\tilde{\sigma}} \rangle$ ,  $P$ -almost surely which implies that the distributions  $P \circ (S^{\hat{\sigma}})^{-1}$  and  $P \circ (S^{\tilde{\sigma}})^{-1}$  are mutually singular.

So, given a family of stochastic processes  $X^P, P \in \mathcal{P}$ , we need to construct a universal process  $X$  which is uniquely defined with respect to all measures at the same time such that  $X = X^P$   $P$ -a.s. for all  $P \in \mathcal{P}$ . Also when defining a stochastic integral  $I_t^P := \int_0^t \eta_s dB_s$  for all  $P \in \mathcal{P}$  simultaneously the same situation arises. Clearly, we can define  $I_t^P$  under each  $P$  in the classical sense. Since  $I_t^P$  may depend on the respective underlying measure  $P$  we are free to redefine the integral outside the support of  $P$ . Thus in order to make things work we need to find a universal integral  $I_t$  satisfying  $I_t = I_t^P$   $P$ -a.s. for all measures  $P \in \mathcal{P}$ .

Let us now come to the introduction of the financial market. All along the paper we consider a financial market  $\mathcal{M}$  consisting of two assets evolving according to

$$\begin{aligned} d\gamma_t &= r\gamma_t dt, \quad \gamma_0 = 1, \\ dS_t &= rS_t dt + S_t dB_t, \quad S_0 = x_0 > 0, \end{aligned} \tag{3}$$

with a constant interest rate  $r \geq 0$ .  $B = (B_t)$  denotes the canonical process which is a G-Brownian motion under  $E_G$  with parameters  $\bar{\sigma} > \underline{\sigma} > 0$ . For the exact definition and construction of the pair  $B$  and  $E_G$  see Appendix A.1. The assumption of strict positive volatility is well accepted in Finance. It is of economic relevance, see also Remark A.5. The asset  $\gamma = (\gamma_t)$  represents a riskless bond as usual. Since  $B = (B_t)$  is a G-Brownian motion,  $S$  is modeled as a geometric G-Brownian motion similarly to the original Black-Scholes model, cf. Black and Scholes (1973), where the stock price is modeled by a classical geometric Brownian motion.

As a consequence, in this market the stock price evolution does not only involve risk modeled by the noise part but also ambiguity about the risk due to the unknown deviation of the process  $B$  from its mean. In terms of Finance this ambiguity in the stock price is called volatility uncertainty. If we choose  $\sigma = \underline{\sigma} = \bar{\sigma}$  we are in the classical Black-Scholes model, see Black and Scholes (1973) or any good textbook in Finance.

**Remark 2.2** Note that the discounted stock price process  $(\gamma_t^{-1} S_t)$  is directly modeled as a symmetric G-martingale with regard to the corresponding G-expectation  $E_G$ . It is a well known fact in Finance that problems like pricing



or hedging contingent claims for instance are handled under a risk-neutral probability measure which leads to the favored situation in which the discounted stock price process is a (local) martingale, cf. Duffie (1992).<sup>13</sup>

The use of G-Brownian motion in order to model the financial market initially leads to a formulation of  $\mathcal{M}$  which is not based on a classical probability space. The representation theorem for G-expectation, see Theorem A.12, establishes a link also to a probabilistic framework. It provides us with a probability space  $(\Omega_T, \mathcal{F}, P)$  and a set of multiple priors  $\mathcal{P}$  such that the following identity holds

$$E_G(X) = \sup_{P \in \mathcal{P}} E^P(X)$$

where  $X$  is any random variable for which the G-expectation can be defined, for instance if  $X : \Omega_T \rightarrow \mathbb{R}$  is bounded and continuous.  $\mathcal{F} = \mathcal{B}(\Omega_T)$  denotes the Borel  $\sigma$ -algebra on the path space  $\Omega_T = C_0([0, T], \mathbb{R})$ . Besides, there exists a process  $W = (W_t)$  which is a classical Brownian motion w.r.t.  $P$ . We can consider the filtration  $(\mathcal{F}_t)$  generated by  $W$ , i.e.,  $\mathcal{F}_t := \sigma\{W_s | 0 \leq s \leq t\} \vee \mathcal{N}$  where  $\mathcal{N}$  denotes the collection of  $P$ -null subsets. Then the set of multiple priors  $\mathcal{P}$  can be constructed as follows.

Let  $\Theta := [\underline{\sigma}, \bar{\sigma}]$ , and  $\mathcal{A}_{0,T}^\Theta$  be the collection of all  $\Theta$ -valued  $(\mathcal{F}_t)$ -adapted processes on  $[0, T]$ . For any  $\theta \in \mathcal{A}_{0,T}^\Theta$  we define  $B_t^{0,\theta} := \int_0^t \theta_s dW_s$  and  $P^\theta$  as the law of  $B^{0,\theta} = \int_0^\cdot \theta_s dW_s$ , i.e.,  $P^\theta = P \circ (B^{0,\theta})^{-1}$ . Then  $\mathcal{P}$  is the closure of  $\{P^\theta | \theta \in \mathcal{A}_{0,T}^\Theta\}$  under the topology of weak convergence.

All along the paper we will consider the induced tuple  $(\Omega_T, \mathcal{F}, (\mathcal{F}_t), W, P)$  together with the set of priors  $\mathcal{P}$  as given. Since  $\mathcal{P}$  represents  $E_G$ , it also represents the volatility uncertainty of the stock price and therefore of  $\mathcal{M}$ . The G-framework utilized in this paper enables the analysis of stochastic processes for all priors of  $\mathcal{P}$  simultaneously. The terminology of “quasi-sure” turns out to be very useful:

A set  $A \in \mathcal{F}$  is called *polar* if  $P(A) = 0$  for all  $P \in \mathcal{P}$ . We say a property holds “*quasi-surely*” (q.s.) if it holds outside a polar set.

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<sup>13</sup>This should also be the case in our ambiguous setting. By modeling the discounted stock price directly as a symmetric G-martingale we do not have to change the sublinear expectation from a subjective to the risk-neutral sublinear expectation and avoid the technical difficulties involved. In an accompanying paper we will give economic verification for this dynamics.

When not stated otherwise all equations are also to be understood in the sense of “quasi-sure”. This means that a property holds almost-surely for all conceivable scenarios.

Next we repeat some useful definitions which are standard in Finance but have to be adapted to this more complex situation. We will need to use the following spaces  $L_G^p(\Omega_T)$ ,  $H_G^p(0, T)$ , and also  $M_G^p(0, T)$ ,  $p \geq 1$ , which denote specific spaces in the G-setting. The first one concerns random variables for which the G-expectation is defined, see Equation (9) in Appendix A.1. The other two are particular spaces of processes for which stochastic integrals with respect to  $B$  or  $\langle B \rangle$ , respectively, can be defined. They are the closure of collections of simple processes similar to the case when the classical Itô integral is constructed, see also Appendix A.2 at pages 34 – 36.

All along the paper we will presume a finite time horizon denoted by  $T > 0$ .

**Definition 2.3** *A trading strategy in the market  $\mathcal{M}$  is an  $(\mathcal{F}_t)$ -adapted vector process  $(\eta, \phi) = (\eta_t, \phi_t)$ ,  $\phi$  a member of  $H_G^1(0, T)$  such that  $(\phi_t S_t) \in H_G^1(0, T)$ .*

*A cumulative consumption process  $C = (C_t)$  is a nonnegative  $(\mathcal{F}_t)$ -adapted process with values in  $L_G^1(\Omega_T)$ , and with increasing, RCLL paths on  $(0, T]$ , and  $C_0 = 0$ ,  $C_T < \infty$  q.s.*

Note that the stock’s price process  $S$  defined by (3) is an element of  $M_G^2(0, T)$  which coincides with  $H_G^2(0, T)$ , see Peng (2010). We impose the so-called self-financing condition, that is, consumption and trading in  $\mathcal{M}$  satisfy

$$V_t := \eta_t \gamma_t + \phi_t S_t = \eta_0 \gamma_0 + \phi_0 S_0 + \int_0^t \eta_u d\gamma_u + \int_0^t \phi_u dS_u - C_t \quad \forall t \leq T \text{ q.s.} \quad (4)$$

where  $V_t$  denotes the value of the trading strategy at time  $t$ .

Sometimes, it is more appropriate to consider instead of a trading strategy a *portfolio process* which presents the proportions of wealth invested in the risky stock.

**Remark 2.4** *A portfolio process  $\pi$  represents proportions of a wealth  $X$  which are invested in the stock within the considered time interval whereas a trading strategy  $(\eta, \phi)$  represents the total numbers of the respective assets the*

agent holds. Clearly, there is a one-to-one correspondence between a portfolio process and a trading strategy as defined above. If we define

$$\phi_t := \frac{X_t \pi_t}{S_t}, \quad \eta_t := \frac{X_t(1 - \pi_t)}{\gamma_t}, \quad \forall t \leq T,$$

$(\eta, \phi)$  constitutes a trading strategy in the sense of equation (4) as long as  $\pi$  constitutes a portfolio process with corresponding wealth process  $X$  as required in Definition 2.6 below.

**Definition 2.5** A portfolio process is an  $(\mathcal{F}_t)$ -adapted real valued process  $\pi = (\pi_t)$  with values in  $L_G^1(\Omega_T)$ .

**Definition 2.6** For a given initial capital  $y$ , a portfolio process  $\pi$ , and a cumulative consumption process  $C$ , consider the wealth equation

$$\begin{aligned} dX_t &= X_t(1 - \pi_t) \frac{d\gamma_t}{\gamma_t} + X_t \pi_t \frac{dS_t}{S_t} - dC_t \\ &= X_t r dt + X_t \pi_t dB_t - dC_t \end{aligned}$$

with initial wealth  $X_0 = y$ . Or equivalently,

$$\gamma_t^{-1} X_t = y - \int_0^t \gamma_u^{-1} dC_u + \int_0^t \gamma_u^{-1} X_u \pi_u dB_u, \quad \forall t \leq T.$$

If this equation has a unique solution  $X = (X_t) := X^{y, \pi, C}$  it is called the wealth process corresponding to the triple  $(y, \pi, C)$ .

In order to have the stochastic integral well defined,  $\int_0^T X_t^2 \pi_t^2 dt < \infty$  must hold quasi-surely and we need to impose the requirement that  $(\pi_t X_t) \in H_G^p(0, T)$ ,  $p \geq 1$ , or  $\in M_G^p(0, T)$ ,  $p \geq 2$ . We incorporate this into the next definition which describes admissible portfolio processes.

**Definition 2.7** A portfolio/consumption process pair  $(\pi, C)$  is called admissible for an initial capital  $y \in \mathbb{R}$  if

(i) the pair obeys the conditions of Definitions 2.3 – 2.6

(ii)  $(\pi_t X_t^{y, \pi, C}) \in H_G^1(0, T)$

(iii) the solution  $X_t^{y,\pi,C}$  satisfies

$$X_t^{y,\pi,C} \geq -L, \quad \forall t \leq T, \text{ q.s.}$$

where  $L$  is a nonnegative random variable in  $L_G^2(\Omega_T)$ .

We then write  $(\pi, C) \in \mathcal{A}(y)$ .

In the above Definitions 2.3 – 2.7 we need to assure that the associated stochastic integrals are well-defined. In particular condition (ii) of Definition 2.7 ensures that the mathematical framework does not collapse by allowing for too many portfolio processes.

The agent is uncertain about the true volatility, therefore, she uses a portfolio strategy which can be performed independently of the realized scenario at the market. Hence, she is able to analyze the corresponding wealth processes with respect to all conceivable market scenarios  $P \in \mathcal{P}$  simultaneously.

These restrictions on the portfolio and consumption processes replace the classical condition of predictable processes. Decisions at some time  $t$  must not utilize information which is revealed subsequently. In our financial setting, the processes have to be members of particular spaces within the G-framework. Based on the construction of these spaces (by means of (viscosity) solutions of PDEs, cf. Appendix A) the portfolio and consumption processes require some kind of regularity, in particular see identity (9) in Appendix A.1. The economic interpretation is that decisions should not react too abruptly and sensitive to revealed information.

### 3 Arbitrage and contingent claims

As usual in financial markets we impose the concept of arbitrage. Due to this more complex framework both of economic and mathematical manner we need a slightly more involved definition of arbitrage.

**Definition 3.1 (Arbitrage in  $\mathcal{M}$ )** *We say there is an arbitrage opportunity in  $\mathcal{M}$  if there exist an initial wealth  $y \leq 0$ , an admissible pair  $(\pi, C) \in \mathcal{A}(y)$  with  $C \equiv 0$  such that at some time  $T > 0$*

$$X_T^{y,\pi,0} \geq 0 \text{ q.s., and} \\ P(X_T^{y,\pi,0} > 0) > 0 \text{ for at least one } P \in \mathcal{P}.$$

If such a strategy in the sense above existed one should pursue this strategy since it would be riskless and in the lucky situation that  $P$  drove the market dynamics one would make a profit with positive probability. It is important to note that in the definition of arbitrage we have to require quasi-sure dominance for the wealth at time  $T$  in order to exclude the risk in all possible scenarios that may occur. So there should not exist a scenario under which there is positive probability that the terminal wealth is less than zero.

**Remark 3.2** *The second condition in the definition of arbitrage is just the negation of  $X_T^{y,\pi,C} \leq 0$  q.s. Hence, combined with the first condition it excludes that  $X_T^{y,\pi,C}$  equals zero quasi-surely.*

*We identify  $(y, \pi, C)$  as an arbitrage if there exists profit with positive probability in at least one scenario even though there does not exist profit with positive probability in many others. Of course, one could also define arbitrage by the requirement that the second condition has to hold for all scenarios, i.e., there existed profit with positive probability in all scenarios. We think that this kind of arbitrage definition is not very reasonable from an economic point of view, see Remark 3.14.*

**Lemma 3.3 (No-arbitrage)** *In the financial market  $\mathcal{M}$  there does not exist any arbitrage opportunity.*

PROOF: Assume there exists an arbitrage opportunity, i.e., there exists some  $y \leq 0$  and a pair  $(\pi, C) \in \mathcal{A}(y)$  with  $C \equiv 0$  such that  $X_T^{y,\pi,0} \geq 0$  quasi-surely for some  $T > 0$ . Then we have  $E_G(X_T^{y,\pi,0}) \geq 0$ . By definition of the wealth process

$$0 \leq E_G(X_T^{y,\pi,0} \gamma_T^{-1}) \leq y + E_G\left(\int_0^T \gamma_t^{-1} X_t^{y,\pi,0} \pi_t dB_t\right) = y$$

since the  $G$ -expectation of an integral with respect to  $G$ -Brownian motion is zero.<sup>14</sup> Hence,  $E_G(X_T^{y,\pi,0} \gamma_T^{-1}) = 0$  which again implies  $X_T^{y,\pi,0} \gamma_T^{-1} = 0$  q.s. Thus,  $(y, \pi, 0)$  is not an arbitrage.  $\square$

In the financial market  $\mathcal{M}$  we want to consider European contingent claims  $H$  with payoff  $H_T$  at maturity time  $T$ . Here,  $H_T$  represents a nonnegative,  $\mathcal{F}_T$ -adapted random variable. All the time we impose the assumption

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<sup>14</sup>Here we used that  $(X_t^{y,\pi,0} \pi_t) \in H_G^1(0, T)$ , cf. condition (ii) of Definition 2.7.

$H_T \in L_G^2(\Omega_T)$ . The price of the claim at time 0 will be denoted by  $H_0$ . In order to find reasonable prices for  $H$  we use the concept of arbitrage. Similar to above we define an arbitrage opportunity in the financial market  $(\mathcal{M}, H)$  consisting of the original market  $\mathcal{M}$  and the contingent claim  $H$ .

**Definition 3.4 (Arbitrage in  $(\mathcal{M}, H)$ )** *There is an arbitrage opportunity in  $(\mathcal{M}, H)$  if there exist an initial wealth  $y \geq 0$  (respectively,  $y \leq 0$ ), an admissible pair  $(\pi, C) \in \mathcal{A}(y)$  and a constant  $a = -1$  (respectively,  $a=1$ ), such that*

$$y + a \cdot H_0 \leq 0$$

*at time 0, and*

$$X_T^{y,\pi,C} + a \cdot H_T \geq 0 \quad q.s., \quad \text{and} \\ P \left( X_T^{y,\pi,C} + a \cdot H_T > 0 \right) > 0 \quad \text{for at least one } P \in \mathcal{P}$$

*at time  $T$ .*

The values  $a = \pm 1$  in Definition 3.4 indicate long or short positions in the claim  $H$ , respectively. This definition of arbitrage is standard in the literature, see Karatzas and Shreve (1998). For the same reasons as before we again require quasi-sure dominance for the wealth at time  $T$  and gain with positive probability for only one possible scenario.

In the following we show the existence of no-arbitrage prices for a claim  $H$  which exclude arbitrage opportunities. Compared to the classical Black-Scholes model there are many no-arbitrage prices for  $H$  in general. We shall see that mostly hedging, or replicating arguments, respectively, fail due to the additional source of uncertainty induced by the G-normal distribution causing the incompleteness of the financial market, see Remark 3.11. Thus in our ambiguous market  $\mathcal{M}$  there generally is either a self-financing portfolio strategy which replicates the European claim nor a risk-free hedge for the claim since the uncertainty represented by the occurring quadratic variation term cannot be eliminated. Only for special claims  $H$  when  $(E_G[H_T|\mathcal{F}_t])$  is a symmetric G-martingale, cf. Remark 3.11, we have  $h_{up} = h_{low}$ .

Clearly, there is only one single G-Brownian motion which occurs in the financial market model. However, due to the representation theorem for G-expectation there are many probability measures involved in  $\mathcal{M}$ , cf. Theorem

A.12. Each measure reflects a specific volatility rate for the stock price. Roughly speaking, these measures induce the incompleteness since only one scenario is being realized and only in this scenario the stock is being traded.

The functional  $E_G$  is just a useful method to control the dynamics by giving upper and lower bounds for European contingent claim prices written on the stock, see Theorem 3.6.

The following classes will matter in our subsequent analysis.

**Definition 3.5** *Given a European contingent claim  $H$  we define the lower hedging class*

$$\mathcal{L} := \{y \geq 0 \mid \exists (\pi, C) \in \mathcal{A}(-y) : X_T^{y, \pi, C} \geq -H_T \text{ q.s.}\}$$

*and the upper hedging class*

$$\mathcal{U} := \{y \geq 0 \mid \exists (\pi, C) \in \mathcal{A}(y) : X_T^{y, \pi, C} \geq H_T \text{ q.s.}\}.$$

*In addition, the lower arbitrage price is defined as*

$$h_{low} := \sup\{y \mid y \in \mathcal{L}\}$$

*and the upper arbitrage price as*

$$h_{up} := \inf\{y \mid y \in \mathcal{U}\}.$$

The main result of this section concerns the lower and upper arbitrage price. It is possible to determine the prices explicitly. We have

**Theorem 3.6** *Given the financial market  $(\mathcal{M}, H)$ . The following identities hold:*

$$\begin{aligned} h_{up} &= E_G(H_T \gamma_T^{-1}) \\ h_{low} &= -E_G(-H_T \gamma_T^{-1}). \end{aligned}$$

Before proving the theorem we establish some results about the hedging classes. As proved in Karatzas and Kou (1996) one can easily show that  $\mathcal{L}$  and  $\mathcal{U}$  are connected intervals. More precisely we have

**Lemma 3.7**  *$y \in \mathcal{L}$  and  $0 \leq z \leq y$  implies  $z \in \mathcal{L}$ . Analogously,  $y \in \mathcal{U}$  and  $z \geq y$  implies  $z \in \mathcal{U}$ .*



The proof uses the idea that one “just consumes immediately the difference of the two initial wealth”. To include the case  $\mathcal{U} = \emptyset$  we define  $\inf \emptyset = \infty$ .

For  $\sigma \in [\underline{\sigma}, \bar{\sigma}]$  let us define the Black–Scholes price of a European contingent claim  $H$

$$u_0^\sigma := E^{P^\sigma}(H_T \gamma_T^{-1})$$

where  $P^\sigma \in \mathcal{P}$  denotes the measure under which  $S$  has constant volatility level  $\sigma$ . As mentioned in Appendix A.1 it is defined by  $P^\sigma := P_0 \circ (X^\sigma)^{-1}$  where  $X_t^\sigma := \int_0^t \sigma dW_u$ . Clearly, due to the dynamics of  $S$ , cf. Equation 3,  $P^\sigma$  is the usual risk neutral probability measure in the Black–Scholes model with fixed volatility rate  $\sigma$ .

Similar as in the case with constraints, see Karatzas and Kou (1996), we can prove the following three lemmata. For this let  $H$  be a given European contingent claim.

**Lemma 3.8** *For any  $\sigma \in [\underline{\sigma}, \bar{\sigma}]$  the following inequality chain holds:*

$$h_{low} \leq u_0^\sigma \leq h_{up}.$$

PROOF: Let  $y \in \mathcal{U}$ . By definition of  $\mathcal{U}$  there exists a pair  $(\pi, C) \in \mathcal{A}(y)$  such that  $X_T^{y, \pi, C} \geq H_T$  q.s. Using the properties of G–expectation as stated in Appendix A.1, in particular Prop. A.11 for the first equality, we obtain for any  $\sigma \in [\underline{\sigma}, \bar{\sigma}]$

$$\begin{aligned} y &= E_G \left( y + \int_0^T \gamma_t^{-1} X_t^{y, \pi, C} \pi_t dB_t \right) \geq E_G \left( y + \int_0^T \gamma_t^{-1} X_t^{y, \pi, C} \pi_t dB_t - \int_0^T \gamma_t^{-1} dC_t \right) \\ &= E_G \left( X_T^{y, \pi, C} \gamma_T^{-1} \right) \geq E_G (H_T \gamma_T^{-1}) = \sup_{P \in \mathcal{P}} E^P (H_T \gamma_T^{-1}) \geq u_0^\sigma. \end{aligned}$$

The first and second inequalities hold due to the monotonicity of  $E_G$ , the second equality holds by the definition of the wealth process and due to  $y \in \mathcal{U}$ , the third equality by the representation theorem for  $E_G$ , cf. Theorem A.12, and the last estimate holds because of  $P^\sigma \in \mathcal{P}$ . Hence,  $h_{up} \geq u_0^\sigma$ .

Similarly, let  $y \in \mathcal{L}$  and  $(\pi, C) \in \mathcal{A}(-y)$  be the corresponding pair such that  $X_T^{-y, \pi, C} \geq -H_T$  q.s. By the same reasoning as above we obtain for any

$$\sigma \in [\underline{\sigma}, \bar{\sigma}]$$

$$\begin{aligned} -y &= E_G \left( -y + \int_0^T \gamma_t^{-1} X_t^{-y, \pi, C} \pi_t dB_t \right) \\ &\geq E_G \left( -y + \int_0^T \gamma_t^{-1} X_t^{-y, \pi, C} \pi_t dB_t - \int_0^T \gamma_t^{-1} dC_t \right) \\ &= E_G \left( X_T^{-y, \pi, C} \gamma_T^{-1} \right) \geq E_G \left( -H_T \gamma_T^{-1} \right) \geq -E^{P^\sigma} \left( H_T \gamma_T^{-1} \right) = -u_0^\sigma \end{aligned}$$

which implies  $y \leq u_0^\sigma$  and the Lemma follows.  $\square$

**Lemma 3.9** *For any price  $H_0 > h_{up}$  there exists an arbitrage opportunity. Also for any price  $H_0 < h_{low}$  there exists an arbitrage opportunity.*

PROOF: We only consider the first case since the argument is similar. Assume  $H_0 > h_{up}$  and let  $y \in (h_{up}, H_0)$ . By definition of  $h_{up}$  we deduce that  $y \in \mathcal{U}$ . Hence there exists a pair  $(\pi, C) \in \mathcal{A}(y)$  with

$$X_T^{y, \pi, C} \geq H_T \quad q.s.$$

and

$$y - H_0 < 0.$$

This implies the existence of arbitrage in the sense of Definition 3.4:

$\exists a > 1$  with  $ay = H_0$ . Then  $(\pi, aC) \in \mathcal{A}(ay)$  and  $X_T^{ay, \pi, aC} = aX_T^{y, \pi, C}$ . Let  $P \in \mathcal{P}$ , w.l.o.g. we may assume  $P(H_T > 0) > 0$ . Due to

$$1 = P(X_T^{y, \pi, C} \geq H_T) \leq P(aX_T^{y, \pi, C} > H_T) + P(X_T^{y, \pi, C} = 0 = H_T)$$

we deduce  $P(X_T^{ay, \pi, aC} > H_T) > 0$ . Hence,  $(ay, \pi, aC)$  constitutes an arbitrage.  $\square$

**Lemma 3.10** *For any  $H_0 \notin \mathcal{L} \cup \mathcal{U}$  the financial market  $(\mathcal{M}, H)$  is arbitrage free.*

PROOF: Assume  $H_0 \notin \mathcal{U}$ ,  $H_0 \notin \mathcal{L}$  and that there exists an arbitrage opportunity in  $(\mathcal{M}, H)$ . We suppose that it satisfies Definition 3.4 for  $a = -1$ . The case  $a = 1$  works similarly.

By definition of arbitrage there exist  $y \geq 0$ ,  $(\pi, C) \in \mathcal{A}(y)$  with

$$y = X_0^{(y, \pi, C)} \leq H_0$$

and

$$X_T^{y,\pi,C} \geq H_T \quad q.s.$$

Hence,  $y \in \mathcal{U}$ , whence  $H_0 \in \mathcal{U}$  by Lemma 3.7. This contradicts our assumption.  $\square$

Now we pass to the proof of Theorem 3.6.

PROOF: Let us begin with the first identity  $h_{up} = E_G(H_T \gamma_T^{-1})$ . As seen in the proof of Lemma 3.8, for any  $y \in \mathcal{U}$  we have  $y \geq E_G(H_T \gamma_T^{-1})$ . Hence,  $h_{up} = \inf\{y | y \in \mathcal{U}\} \geq E_G(H_T \gamma_T^{-1})$ .

To show the opposite inequality define the G-martingale  $M$  by

$$M_t := E_G(H_T \gamma_T^{-1} | \mathcal{F}_t) \quad \forall t \leq T.$$

By the martingale representation theorem (Song (2010b)), see Theorem A.19, there exist  $z \in H_G^1(0, T)$  and a continuous, increasing process  $K = (K_t)$  with  $K_T \in L_G^1(\Omega_T)$  such that for any  $t \leq T$

$$M_t = E_G(H_T \gamma_T^{-1}) + \int_0^t z_s dB_s - K_t \quad q.s.$$

For any  $t \leq T$  we set  $y = E_G(H_T \gamma_T^{-1}) \geq 0$ ,  $X_t \pi_t = z_t \gamma_t \in H_G^1(0, T)$ , and  $C_t = \int_0^t \gamma_s dK_s \in L_G^1(\Omega_T)$ . Then the induced wealth process  $X^{y,\pi,C}$  satisfies for any  $t \leq T$

$$\gamma_t^{-1} X_t^{y,\pi,C} = y + \int_0^t X_s^{y,\pi,C} \pi_s \gamma_s^{-1} dB_s - \int_0^t \gamma_s^{-1} dC_s = M_t.$$

$C$  obeys the conditions of a cumulative consumption process in the sense of Definition 2.3 due to the properties of  $K$ . Because of  $\gamma_t^{-1} X_t^{y,\pi,C} = M_t \geq 0 \quad \forall t \leq T$  the wealth process is bounded from below, whence  $(\pi, C)$  is admissible for  $y$ .

As  $X_T^{y,\pi,C} = \gamma_T M_T = H_T$  quasi-surely we have  $y = E_G(H_T \gamma_T^{-1}) \in \mathcal{U}$ . Due to the definition of  $\mathcal{U}$  we conclude  $h_{up} \leq E_G(H_T \gamma_T^{-1})$ .

The proof for the second identity is similar. Again, using the proof of Lemma 3.8 we obtain  $y \leq -E_G(-H_T \gamma_T^{-1})$  for any  $y \in \mathcal{L}$  and therefore  $h_{low} \leq -E_G(-H_T \gamma_T^{-1})$ .

To obtain  $h_{low} \geq -E_G(-H_T\gamma_T^{-1})$  we again define a G-martingale  $M$  by

$$M_t = E_G(-H_T\gamma_T^{-1}|\mathcal{F}_t) \quad \forall t \leq T.$$

The remaining part is almost a copy of above. Again by the martingale representation theorem (Song (2010b)) there exist  $z \in H_G^1(0, T)$  and a continuous, increasing process  $K = (K_t)$  with  $K_T \in L_G^1(\Omega_T)$  such that for any  $t \leq T$

$$M_t = E_G(-H_T\gamma_T^{-1}) + \int_0^t z_s dB_s - K_t \quad q.s.$$

As above, for any  $t \leq T$  we set  $-y = E_G(-H_T\gamma_T^{-1}) \geq 0$ ,  $X_t\pi_t = z_t\gamma_t \in H_G^1(0, T)$ , and  $C_t = \int_0^t \gamma_s dK_s \in L_G^1(\Omega_T)$ . Then the induced wealth process  $X^{-y,\pi,C}$  satisfies for all  $t \leq T$

$$\gamma_t^{-1} X_t^{-y,\pi,C} = -y + \int_0^t X_s^{-y,\pi,C} \pi_s \gamma_s^{-1} dB_s - \int_0^t \gamma_s^{-1} dC_s = M_t.$$

Again  $C$  obeys the conditions of a cumulative consumption process due to the properties of  $K$ . Furthermore, for any  $t \leq T$

$$\gamma_t^{-1} X_t^{-y,\pi,C} = E_G(-H_T\gamma_T^{-1}|\mathcal{F}_t) \geq E_G(-H_T|\mathcal{F}_t)$$

which is bounded from below in the sense of item (iii) in Definition 2.7 since  $-H_T \in L_G^2(\Omega_T)$ . Hence the wealth process is bounded from below. Consequently,  $(\pi, C)$  is admissible for  $-y$ .

As  $X_T^{-y,\pi,C} = \gamma_T M_T = -H_T$  quasi-surely we have  $y = -E_G(-H_T\gamma_T^{-1}) \in \mathcal{L}$ . Due to the definition of  $\mathcal{L}$  we conclude  $h_{low} \geq -E_G(-H_T\gamma_T^{-1})$  which finishes the proof.  $\square$

**Remark 3.11** By the last theorem we have  $h_{low} \neq h_{up}$  in general since  $E_G$  is a sublinear expectation. This implies that the market is incomplete meaning that not all claims can be hedged perfectly. Thus in general, there are many no-arbitrage prices for  $H$ . We always have  $h_{low} \neq h_{up}$  as long as  $(E_G[H_T\gamma_T^{-1}|\mathcal{F}_t])$  is not a symmetric G-martingale. In the other case, the process  $K$  is identically equal to zero, cf. Remark A.18, implying that  $(E_G[H_T\gamma_T^{-1}|\mathcal{F}_t])$  is symmetric and  $H_T$  can be hedged perfectly due to Theorem A.19 and Remark A.18. As it is being showed in Section 4, if  $H$  for instance is the usual European call or put option this is only the case if  $\underline{\sigma} = \bar{\sigma}$  which again implies that  $E_G$  becomes the classical expectation.

**Remark 3.12** *Again under the presumption of  $h_{low} \neq h_{up}$  it is not clear a priori whether a claim's price  $H_0$  equal to  $h_{up}$  or  $h_{low}$  induces an arbitrage opportunity or not. In the setting of Karatzas and Kou (1996) there may be situations where there is no arbitrage, while in others there may be arbitrage. For instance, if  $H_0 = h_{up} \in \mathcal{U}$  and  $C_T > 0$  a.s., then this consumption can be viewed as kind of arbitrage opportunity (see Karatzas and Kou (1996)). The agent consumes along the way, and ends up with terminal wealth  $H_T$  almost surely.*

*As seen in the proof of Theorem 3.6, in our setting we always have  $h_{up} \in \mathcal{U}$  and  $h_{low} \in \mathcal{L}$ . We shall see that due to our definition of arbitrage –  $P\left(X_T^{y,\pi,C} - aH_T > 0\right) > 0$  only has to hold for one  $P \in \mathcal{P}$  – we have that a price  $H_0 = h_{up}$  or  $H_0 = h_{low}$  induces arbitrage in  $(\mathcal{M}, H)$  in the sense of Definition 3.4.*

**Corollary 3.13** *For any price  $H_0 \in (h_{low}, h_{up}) \neq \emptyset$  of a European contingent claim at time zero there does not exist any arbitrage opportunity in  $(\mathcal{M}, H)$ . For any price  $H_0 \notin (h_{low}, h_{up}) \neq \emptyset$  there does exist arbitrage in the market.*

PROOF: The first part directly follows from Lemma 3.10. From Lemma 3.9 we know that  $H_0 \notin [h_{low}, h_{up}]$  implies the existence of an arbitrage opportunity. Therefore we only have to show that  $H_0 = h_{up}$  and  $H_0 = h_{low}$  admits an arbitrage opportunity.

We only treat the case  $H_0 = h_{up}$ , the second case is analogue. Comparing the proof of Theorem 3.6, for  $y = E_G(H_T \gamma_T^{-1})$  there exists a pair  $(\pi, C) \in \mathcal{A}(y)$  such that

$$\gamma_T^{-1} X_T^{y,\pi,C} = y + \int_0^T X_s^{y,\pi,C} \pi_s \gamma_s^{-1} dB_s - \int_0^T \gamma_s^{-1} dC_s = H_T \gamma_T^{-1} \quad q.s.$$

We had  $K_T = \int_0^T \gamma_s^{-1} dC_s$  where  $K$  was an increasing, continuous process with  $E_G(-K_T) = 0$ . Hence we can select  $P \in \mathcal{P}$  such that  $E_P(-K_T) < 0$ , see also Remark 3.11. Then the pair  $(\pi, 0) \in \mathcal{A}(y)$  satisfies

$$E^P(\gamma_T^{-1} X_T^{y,\pi,0}) > E^P(\gamma_T^{-1} X_T^{y,\pi,C}) = E^P(H_T \gamma_T^{-1}).$$

Thus,  $P(X_T^{y,\pi,0} > H_T) > 0$  and we conclude that  $(\pi, 0) \in \mathcal{A}(y)$  constitutes an arbitrage. So, possibly the agent may consume along the way, and ends up with wealth  $H_T$  quasi-surely.  $\square$

**Remark 3.14** *Note that the second statement of the corollary heavily depends on the definition of arbitrage. Under the assumption of  $h_{low} \neq h_{up}$  it states that if  $H_0$  is equal to one of the bounds  $h_{up}$  or  $h_{low}$  there exists arbitrage in the sense of Definition 3.4.*

*Coming back to the discussion about the definition of arbitrage started in Remark 3.2 the proofs of the corollary and Theorem 3.6 also imply that if we required the last condition in Definition 3.4 to be true for all scenarios  $P \in \mathcal{P}$  then  $H_0$  equal to one of the bounds would not induce arbitrage in this new sense. Hence,  $h_{up}$  and  $h_{low}$  would be reasonable prices for the claim.*

*However, there would exist profit with positive probability in many scenarios. Only the scenarios  $P \in \mathcal{P}$  that satisfy  $E^P(-K_T) = 0$  would not provide profit with positive probability. Thus, all  $P \in \mathcal{P}$  not being maximizer of  $\sup_{P \in \mathcal{P}} E^P(-K_T)$  would induce arbitrage in the classical sense when only one probability measure is involved.*

*From our point of view such a situation should be identified as arbitrage which therefore supports our definition of arbitrage in 3.1 and 3.4.*

*Additionally, even though our arbitrage definition requires profit with positive probability for only one scenario it is simultaneously satisfied for all  $P \in \mathcal{P}$  which are not maximizer of  $\sup_{P \in \mathcal{P}} E^P(-K_T)$ .*

Based on the corollary we call  $(h_{low}, h_{up}) \neq \emptyset$  the arbitrage free interval. In the case where a more explicit martingale representation theorem for  $(E_G[\gamma_T^{-1} H_T | \mathcal{F}_t])$  holds, see Hu and Peng (2010), we obtain a more explicit form for the consumption process  $C$ . In particular in the Markovian setting where  $H_T = \Phi(S_T)$  for some Lipschitz function  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  we can give more structural details about the bounds  $h_{up}$  and  $h_{low}$ . We investigate this issue in the following section.

## 4 The Markovian setting

We consider the same financial market  $\mathcal{M}$  as before and restrict ourselves to European contingent claims  $H$  which have the form  $H_T = \Phi(S_T)$  for some Lipschitz function  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ .

We will use a nonlinear Feynman–Kac formula established in Peng (2010). For this issue let us rewrite the dynamics of  $S$  in (3) as

$$dS_u^{t,x} = rS_u^{t,x} du + S_u^{t,x} dB_u, \quad u \in [t, T], \quad S_t^{t,x} = x > 0.$$

Similar as the lower and upper arbitrage prices at time 0 we define the lower and upper arbitrage prices at time  $t \in [0, T]$ ,  $h_{low}^t(x)$  and  $h_{up}^t(x)$ . We use the variable  $x$  just to indicate that the stock at a considered time  $t$  is at level  $x$ , i.e.,  $S_t = x$ .

The following theorem is an extension of Theorem 3.6. It establishes the connection of the lower and upper arbitrage prices with solutions of partial differential equations.

**Theorem 4.1** *Given a European contingent claim  $H = \Phi(S_T)$  the upper arbitrage price  $h_{up}^t(x)$  is given by  $u(t, x)$  where  $u : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is the unique solution of the PDE*

$$\partial_t u + rx\partial_x u + G(x^2\partial_{xx}u) = ru, \quad u(T, x) = \Phi(x). \quad (5)$$

*An explicit representation for the corresponding trading strategy in the stock and the cumulative consumption process is given by*

$$\begin{aligned} \phi_t &= \partial_x u(t, S_t) \quad \forall t \in [0, T], \\ C_t &= -\frac{1}{2} \int_0^t \partial_{xx} u(s, S_s) S_s^2 d\langle B \rangle_s + \int_0^t G(\partial_{xx} u(s, S_s)) S_s^2 ds \quad \forall t \in [0, T]. \end{aligned}$$

*Similarly, the lower arbitrage price  $h_{low}^t(x)$  is given by  $-\underline{u}(t, x)$  where  $\underline{u} : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}$  also solves (5) but with terminal condition  $\underline{u}(T, x) = -\Phi(x) \forall x \in \mathbb{R}_+$ . Also, the analog expressions hold true for the corresponding trading strategy and cumulative consumption process.*

The PDE in (5) is called Black–Scholes–Barenblatt equation. It is also established in Avellaneda, Levy, and Paras (1995).

Before passing to the proof let us consider the BSDE

$$Y_s^{t,x} = E_G \left( \Phi(S_T^{t,x}) + \int_s^T f(S_r^{t,x}, Y_r^{t,x}) dr \middle| \mathcal{F}_s \right), \quad s \in [t, T],$$

where  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a given Lipschitz function. Since the BSDE has a unique solution, see Peng (2010), we can define a function  $u : [0, T] \times \mathbb{R}_+ \rightarrow \mathbb{R}$  by  $u(t, x) := Y_t^{t,x}$ ,  $(t, x) \in [0, T] \times \mathbb{R}_+$ . Based on a nonlinear version of the Feynman–Kac formula, see Peng (2010), the function  $u$  is a viscosity solution of the following PDE

$$\partial_t u + rx\partial_x u + G(x^2\partial_{xx}u) + f(x, u) = 0, \quad u(T, x) = \Phi(x). \quad (6)$$



Now we come to the proof of Theorem 4.1.

PROOF: It is enough just to treat the upper arbitrage price. For that purpose define the function

$$\hat{u}(t, x) := E_G \left( \Phi(S_T^{t,x}) \gamma_T^{-1} \right).$$

By arguing as above  $\hat{u}$  solves the PDE in (6) for  $f \equiv 0$ . Since the function  $G$  is non-degenerate  $\hat{u}$  even becomes a classical  $C^{1,2}$ -solution, see Remark A.5 or page 19 in Peng (2010). Therefore, together with Itô's formula (Theorem 5.4 in Li and Peng (2009)) we obtain

$$\begin{aligned} & \hat{u}(t, S_t^{0,x}) - \hat{u}(0, x) \\ &= \int_0^t \partial_t \hat{u}(s, S_s^{0,x}) + r S_s^{0,x} \partial_x \hat{u}(s, S_s^{0,x}) ds \\ &+ \int_0^t S_s \partial_x \hat{u}(s, S_s^{0,x}) dB_s + \int_0^t \frac{1}{2} S_s^2 \partial_{xx} \hat{u}(s, S_s^{0,x}) d\langle B \rangle_s \\ &\stackrel{(6)}{=} \int_0^t S_s \partial_x \hat{u}(s, S_s^{0,x}) dB_s \\ &+ \underbrace{\frac{1}{2} \int_0^t S_s^2 \partial_{xx} \hat{u}(s, S_s^{0,x}) d\langle B \rangle_s - \int_0^t S_s^2 G(\partial_{xx} \hat{u}(s, S_s^{0,x})) ds}_{-K_t = -\int_0^t \gamma_s^{-1} dC_s}. \end{aligned}$$

Next consider the function

$$\tilde{u}(t, x) := \gamma_t \hat{u}(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}_+.$$

As in Theorem 3.6 for  $t = 0$  we can deduce that  $\tilde{u}(t, x) = h_{up}^t(x) \forall (t, x) \in [0, T] \times \mathbb{R}_+$ . In addition, one easily checks that  $\tilde{u}$  is a solution of the PDE in (5). Also the function  $u$  defined by

$$u(t, x) := Y_t^{t,x} = E_G \left( \Phi(S_T^{t,x}) - \int_t^T r Y_s^{t,x} ds \middle| \mathcal{F}_t \right) \quad \forall (t, x) \in [0, T] \times \mathbb{R}_+$$

solves the PDE in (5) due to the nonlinear Feynman-Kac formula since  $f(x, y) = -ry$ . By uniqueness of the solution in (5), see Ishii and Lions (1990) ( $f$  is obviously bounded in  $x$ ), we conclude that  $\tilde{u} = u$ . Hence,  $u(t, x) = E_G(\Phi(S_T^{t,x}) \gamma_{T-t}^{-1}) = h_t^{up}(x) \forall (t, x) \in [0, T] \times \mathbb{R}_+$  and it uniquely solves the PDE.

The explicit expressions for the trading strategy  $\phi$  and the cumulative consumption process  $C$  follow from the calculations above for  $\hat{u}$  and the identity  $\hat{u}(t, x) = u(t, x)\gamma_t^{-1}$ .

Comparing the proof of Theorem 3.6, using its notations and Remark 2.4, we obtain  $z_t = S_t^{0,x} \partial_x \hat{u}(t, S_t^{0,x}) = \phi_t S_t^{0,x} \gamma_t^{-1}$ . Hence,  $\phi_t = \partial_x u(t, S_t) \forall t \in [0, T]$ .

Similarly we derive

$$C_t = \int_0^t \gamma_s dK_s = -\frac{1}{2} \int_0^t S_s^2 \partial_{xx} u(s, S_s) d\langle B \rangle_s + \int_0^t S_s^2 G(\partial_{xx} u(s, S_s)) ds.$$

□

Due to Theorem 4.1 the functions  $u(t, x) = h_{up}^t(x)$  and  $\underline{u}(t, x) = -h_{low}^t(x)$  can be characterized as the unique solutions of the Black–Scholes–Barrenblatt equation. In the case of  $\Phi$  being a convex or concave function, respectively, the PDE in (5) simplifies significantly. Due to the following result it just becomes the classical Black–Scholes PDE in (7) for a certain constant volatility level.

**Lemma 4.2** *1. If  $\Phi$  is convex  $u(t, \cdot)$  is convex for any  $t \leq T$ .  
2. If  $\Phi$  is concave  $u(t, \cdot)$  is concave for any  $t \leq T$ . Analogously, if  $\Phi$  is convex  $\underline{u}(t, \cdot)$  is concave for any  $t \leq T$ . If  $\Phi$  is concave  $\underline{u}(t, \cdot)$  is convex for any  $t \leq T$ .*

**PROOF:** Again we only need to consider the upper arbitrage price. It is determined by the function  $u(t, x) = E_G(\Phi(S_T^{t,x})\gamma_{T-t}^{-1}) \forall (t, x) \in [0, T] \times \mathbb{R}_+$ .

Firstly, let  $\Phi$  be convex,  $t \in [0, T)$ , and  $x, y \in \mathbb{R}_+$ . Then we have for any  $\alpha \in [0, 1]$

$$\begin{aligned} u(t, \alpha x + (1 - \alpha)y) &= E_G \left[ \Phi \left( S_T^{t, \alpha x + (1 - \alpha)y} \right) e^{-r(T-t)} \right] \\ &= E_G \left[ \Phi \left( (\alpha x + (1 - \alpha)y) e^{r(T-t) - \frac{1}{2} \langle B \rangle_{T-t} + B_{T-t}} \right) e^{-r(T-t)} \right] \\ &\leq E_G \left[ \alpha \Phi \left( x e^{r(T-t) - \frac{1}{2} \langle B \rangle_{T-t} + B_{T-t}} \right) + (1 - \alpha) \Phi \left( y e^{r(T-t) - \frac{1}{2} \langle B \rangle_{T-t} + B_{T-t}} \right) \right] e^{-r(T-t)} \\ &\leq E_G \left[ \alpha \Phi \left( x e^{r(T-t) - \frac{1}{2} \langle B \rangle_{T-t} + B_{T-t}} \right) e^{-r(T-t)} \right] \\ &\quad + E_G \left[ (1 - \alpha) \Phi \left( y e^{r(T-t) - \frac{1}{2} \langle B \rangle_{T-t} + B_{T-t}} \right) e^{-r(T-t)} \right] \\ &= \alpha E_G \left[ \Phi \left( S_T^{t,x} \right) e^{-r(T-t)} \right] + (1 - \alpha) E_G \left[ \Phi \left( S_T^{t,y} \right) e^{-r(T-t)} \right] \\ &= \alpha u(t, x) + (1 - \alpha) u(t, y) \end{aligned}$$

where we used the convexity of  $\Phi$ , the monotonicity of  $E_G$  and in the second inequality the sublinearity of  $E_G$ . Thus,  $u(t, \cdot)$  is convex for all  $t \in [0, T]$ .

Secondly, let  $\Phi$  be concave. Define for any  $(t, x) \in [0, T] \times \mathbb{R}_+$

$$v(t, x) := E^P \left[ \Phi \left( \tilde{S}_T^{t,x} \right) e^{-r(T-t)} \right]$$

where

$$d\tilde{S}_s^{t,x} = r\tilde{S}_s^{t,x}ds + \underline{\sigma}\tilde{S}_s^{t,x}dW_s, \quad s \in [t, T], \quad \tilde{S}_t^{t,x} = x.$$

Remember that  $W = (W_t)$  is a classical Brownian motion under  $P$ . Then by the classical Feynman–Kac formula  $v$  solves the Black–Scholes PDE in (7) with  $\bar{\sigma}$  replaced by  $\underline{\sigma}$ .

Since  $E^P$  is linear it is straightforward to show that  $v(t, \cdot)$  is concave for any  $t \in [0, T]$ . As a consequence,  $v$  also solves (5). By uniqueness we conclude  $v = u$ . Hence,  $u(t, \cdot)$  is concave for any  $t \in [0, T]$ .  $\square$

As a consequence we have the following corollary.

**Corollary 4.3** *If  $\Phi$  is convex  $h_{up}^0(x) = E^{P^{\bar{\sigma}}}(\Phi(S_T^{0,x})\gamma_T^{-1})$  and*

$$u(t, x) := E_G(\Phi(S_T^{t,x})\gamma_{T-t}^{-1}) = E^{P^{\bar{\sigma}}}(\Phi(S_T^{t,x})\gamma_{T-t}^{-1})$$

*solves the Black–Scholes PDE*

$$\partial_t u + rx\partial_x u + \frac{1}{2}\bar{\sigma}^2 x^2 \partial_{xx} u = ru, \quad u(T, x) = \Phi(x). \quad (7)$$

*If  $\Phi$  is concave  $h_{up}^0(x) = E^{P^{\underline{\sigma}}}(\Phi(S_T^{0,x})\gamma_T^{-1})$  and*

$$u(t, x) := E_G(\Phi(S_T^{t,x})\gamma_{T-t}^{-1}) = E^{P^{\underline{\sigma}}}(\Phi(S_T^{t,x})\gamma_{T-t}^{-1})$$

*solves the PDE in (7) with  $\underline{\sigma}$  replacing  $\bar{\sigma}$ .*

*Clearly, an analogue result holds for the lower arbitrage price  $h_{low}$ , or terminal condition  $\underline{u}(T, x) = -\Phi(x)$ , respectively.*

PROOF: The result directly follows from Theorem 4.1 and Lemma 4.2.  $\square$

**Example 4.4 (European call option)** Consider for  $K > 0$  the function  $\Phi(x) = (x - K)^+$  which represents the payoff of an European call option. Since  $\Phi$  is convex, and  $-\Phi$  concave, we can deduce by means of the last corollary

$$\begin{aligned} h_{up}^0(x) &= E^{P^{\bar{\sigma}}}((S_T^{0,x} - K)^+ \gamma_T^{-1}), \\ h_{low}^0(x) &= -E^{P^{\underline{\sigma}}}(-(S_T^{0,x} - K)^+ \gamma_T^{-1}). \end{aligned}$$

Furthermore, the function

$$u(t, x) := E^{P^{\bar{\sigma}}}((S_T^{t,x} - K)^+ \gamma_{T-t}^{-1}), \quad (t, x) \in [0, T] \times \mathbb{R}_+,$$

solves the PDE in (7). The function

$$\underline{u}(t, x) := E^{P^{\underline{\sigma}}}(-(S_T^{t,x} - K)^+ \gamma_{T-t}^{-1}), \quad (t, x) \in [0, T] \times \mathbb{R}_+,$$

solves Equation (7) with  $\bar{\sigma}$  replaced by  $\underline{\sigma}$  and boundary condition  $\underline{u}(T, x) = -(x - K)^+ \forall x \in \mathbb{R}_+$ .

If  $\Phi$  exhibits mixed convexity/concavity behavior meaning that, for instance, there exists an  $x^* \in \mathbb{R}_+$  such that  $\Phi \upharpoonright_{[0, x^*]}$  is convex whereas  $\Phi \upharpoonright_{[x^*, \infty]}$  is concave, the situation is much more involved.

For instance in the case when  $\Phi$  represents a bullish call spread as considered in Avellaneda, Levy, and Paras (1995) the worst-case volatility will switch between the volatility bounds  $\underline{\sigma}$  and  $\bar{\sigma}$  at some threshold  $\bar{x}(t)$ . The  $t$  indicates the time dependence of the threshold. This fact can be verified by solving the PDE in (5) numerically, see Avellaneda, Levy, and Paras (1995).

Clearly, the evaluation of  $\Phi$  becomes economic relevant when  $\Phi$  represents complex derivatives or a whole portfolio which combines long and short positions. Pricing the whole portfolio is more efficient than pricing the single positions separately and leads to more reasonable results for the no-arbitrage bounds since the bounds are closer based on the subadditivity of  $E_G$ . Numerical methods for solving the Black-Scholes-Barenblatt PDE in (5) can be found in Meyer (2004).

## 5 Conclusion

We present a general framework in mathematical finance in order to deal with model risk caused by volatility uncertainty. This encompasses the extension

of terminology widely used in Finance like portfolio strategy, consumption process, arbitrage prices and the concept of no-arbitrage. It is being modified to a quasi-sure analysis framework resulting from the presence of volatility uncertainty.

Our setting does not involve any reference measure and hence does not exclude any economic interesting model a priori. We consider a stock price modeled by a geometric G-Brownian motion which features volatility uncertainty based on the structure of a G-Brownian motion. In this ambiguous financial setting we examine the pricing and hedging of European contingent claims. The “G-framework” summarized in Peng (2010) gives us a meaningful and appropriate mathematical setting. By means of a slightly new concept of no-arbitrage we establish detailed results which provide a better economic understanding of financial markets under volatility uncertainty.

The current paper may form the basis for examining economic relevant questions in the presence of volatility uncertainty in the sense that it extends important notions in Finance and shows how to control. Concrete examples are problems like hedging under constraints (cf. Karatzas and Kou (1996)) and portfolio optimization (cf. Merton (1990)). A natural step is to extend above results to American contingent claims and then, for instance, consider entry decisions of a firm in the sense of irreversible investments as in Nishimura and Ozaki (2007) who solved the problem in the presence of drift uncertainty.

By the natural properties of sublinear expectation any sublinear expectation induces a coherent risk measure, see Peng (2010). G-expectation may appear as a natural candidate to measure model risk. In this context one might also imagine many concrete applications in Finance.

## A Sublinear expectations

We depict notions and preliminaries in the theory of sublinear expectation and related G-Brownian motion. This includes the definition of G-expectation, introduction to Itô calculus with G-Brownian motion and important results concerning the representation of G-expectation and G-martingales. We do not express definitions and results in their most generality. Our task rather is to present it in a manner of which we used it in the previous sections. More details can be found in Peng (2010) and Li and Peng (2009).

We also restrict ourselves to the one-dimensional case. However, everything also holds in the  $d$ -dimensional case. Also the financial market model can be extended to  $d$  risky assets using a  $d$ -dimensional G-Brownian motion as it is done in classical financial markets with Brownian motion.

## A.1 Sublinear expectation, G-Brownian motion and G-expectation

**Definition A.1** Let  $\Omega \neq \emptyset$  be a given set. Let  $\mathcal{H}$  be a linear space of real valued functions defined on  $\Omega$  with  $c \in \mathcal{H}$  for all constants  $c$  and  $|X| \in \mathcal{H}$  if  $X \in \mathcal{H}$ . ( $\mathcal{H}$  can be considered as the space of random variables.) A sublinear expectation  $\hat{E}$  on  $\mathcal{H}$  is a functional  $\hat{E} : \mathcal{H} \rightarrow \mathbb{R}$  satisfying the following properties: For any  $X, Y \in \mathcal{H}$  we have

(a) *Monotonicity:* If  $X \geq Y$  then  $\hat{E}(X) \geq \hat{E}(Y)$ .

(b) *Constant preserving:*  $\hat{E}(c) = c$ .

(c) *Sub-additivity:*  $\hat{E}(X + Y) \leq \hat{E}(X) + \hat{E}(Y)$ .

(d) *Positive homogeneity:*  $\hat{E}(\lambda X) = \lambda \hat{E}(X) \quad \forall \lambda \geq 0$ .

The triple  $(\Omega, \mathcal{H}, \hat{E})$  is called a sublinear expectation space.

Property (c) is also called self-domination. It is equivalent to  $\hat{E}(X) - \hat{E}(Y) \leq \hat{E}(X - Y)$ . Property (c) together with (d) is called sublinearity. It implies convexity:

$$\hat{E}(\lambda X + (1 - \lambda)Y) \leq \lambda \hat{E}(X) + (1 - \lambda)\hat{E}(Y) \text{ for any } \lambda \in [0, 1].$$

The properties (b) and (c) imply cash translatability:

$$\hat{E}(X + c) = \hat{E}(X) + c \text{ for any } c \in \mathbb{R}.$$

The space  $C_{l,Lip}(\mathbb{R}^n)$ , where  $n \geq 1$  is an integer, plays an important role. It is the space of all real-valued continuous functions  $\varphi$  defined on  $\mathbb{R}^n$  such that  $|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y| \quad \forall x, y \in \mathbb{R}^n$ . Here  $k$  is an integer depending on  $\varphi$ .

**Definition A.2** In a sublinear expectation space  $(\Omega, \mathcal{H}, \hat{E})$  a random variable  $Y \in \mathcal{H}$  is said to be independent from another random variable  $X \in \mathcal{H}$  under  $\hat{E}$  if for any test function  $\varphi \in C_{l,Lip}(\mathbb{R}^2)$  we have

$$\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]_{x=X}].$$

**Definition A.3** Let  $X_1$  and  $X_2$  be two random variables defined on sublinear expectation spaces  $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$  and  $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$ , respectively. They are called identically distributed, denoted by  $X_1 \sim X_2$ , if

$$\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)] \quad \forall \varphi \in C_{l,Lip}(\mathbb{R}).$$

We call  $\bar{X}$  an independent copy of  $X$  if  $\bar{X} \sim X$  and  $\bar{X}$  is independent from  $X$ .

**Definition A.4 (G-normal distribution)** A random variable  $X$  on a sublinear expectation space  $(\Omega, \mathcal{H}, \hat{E})$  is called (centralized) G-normal distributed if for any  $a, b \geq 0$

$$aX + b\bar{X} \sim \sqrt{a^2 + b^2}X$$

where  $\bar{X}$  is an independent copy of  $X$ . The letter  $G$  denotes the function

$$G(y) := \frac{1}{2} \hat{E}[yX^2] : \mathbb{R} \rightarrow \mathbb{R}.$$

Note that  $X$  has no mean-uncertainty, i.e., one can show that  $\hat{E}(X) = \hat{E}(-X) = 0$ . Furthermore, the following important identity holds

$$G(y) = \frac{1}{2} \bar{\sigma}^2 y^+ - \frac{1}{2} \underline{\sigma}^2 y^-$$

with  $\underline{\sigma}^2 := -\hat{E}(-X^2)$  and  $\bar{\sigma}^2 := \hat{E}(X^2)$ . We write  $X$  is  $N(\{0\} \times [\underline{\sigma}^2, \bar{\sigma}^2])$  distributed. Therefore we sometimes say that G-normal distribution is characterized by the parameters  $0 < \underline{\sigma} \leq \bar{\sigma}$ .

**Remark A.5** All along the paper we assume  $\underline{\sigma} > 0$ . From an economic point of view this assumption is quite reasonable. In Finance, volatility is always assumed to be greater zero. A volatility equal to zero would induce arbitrage.

The G-framework also works without this condition. But based on this assumption we get along in our paper without the notion of viscosity solution. Our assumption ensures that the function  $G$  is non-degenerate and therefore all involved PDEs induced by the G-normal distribution, cf. Equation (8), become classical  $C^{1,2}$ -solutions, see page 19 in Peng (2010).



**Remark A.6** The random variable  $X$  defined in A.4 is also characterized by the following parabolic partial differential equation (PDE for short) defined on  $[0, T] \times \mathbb{R}$ :

For any  $\varphi \in C_{l,Lip}(\mathbb{R})$  define  $u(t, x) := \hat{E}[\varphi(x + \sqrt{t}X)]$ , then  $u$  is the unique (viscosity) solution of

$$\partial_t u - G(\partial_{xx} u) = 0, \quad u(0, \cdot) = \varphi(\cdot). \quad (8)$$

The PDE is called a  $G$ -equation.

**Definition A.7** Let  $(\Omega, \mathcal{H}, \hat{E})$  be a sublinear expectation space.  $(X_t)_{t \geq 0}$  is called a stochastic process if  $X_t$  is a random variable in  $\mathcal{H}$  for each  $t \geq 0$ .

**Definition A.8 (G-Brownian motion)** A process  $(B_t)_{t \geq 0}$  on a sublinear expectation space  $(\Omega, \mathcal{H}, \hat{E})$  is called a  $G$ -Brownian motion if the following properties are satisfied:

- (i)  $B_0 = 0$ .
- (ii) For each  $t, s \geq 0$  the increment  $B_{t+s} - B_t$  is  $N(\{0\} \times [\underline{\sigma}^2 s, \bar{\sigma}^2 s])$  distributed and independent from  $(B_{t_1}, B_{t_2}, \dots, B_{t_n})$  for each  $n \in \mathbb{N}$ ,  $0 \leq t_1 \leq \dots \leq t_n \leq t$ .

Condition (ii) can be replaced by the following three conditions giving a characterization of  $G$ -Brownian motion:

- (i) For each  $t, s \geq 0$ :  $B_{t+s} - B_t \sim B_t$  and  $\hat{E}(|B_t|^3) \rightarrow 0$  as  $t \rightarrow 0$ .
- (ii) The increment  $B_{t+s} - B_t$  is independent from  $(B_{t_1}, B_{t_2}, \dots, B_{t_n})$  for each  $n \in \mathbb{N}$  and  $0 \leq t_1 \leq \dots \leq t_n \leq t$ .
- (iii)  $\hat{E}(B_t) = -\hat{E}(-B_t) = 0 \quad \forall t \geq 0$ .

For each  $t_0 > 0$  we have that  $(B_{t+t_0} - B_{t_0})_{t \geq 0}$  again is a  $G$ -Brownian motion.

Let us briefly depict the construction of  $G$ -expectation and its corresponding  $G$ -Brownian motion. As in the previous sections we fix a time horizon  $T > 0$  and set  $\Omega_T = C_0([0, T], \mathbb{R})$  – the space of all real-valued

continuous paths starting at zero. We will consider the canonical process  $B_t(\omega) := \omega_t, t \leq T, \omega \in \Omega$ . We define

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \dots, B_{t_n}) | n \in \mathbb{N}, t_1, \dots, t_n \in [0, T], \varphi \in C_{l,Lip}(\mathbb{R}^n)\}.$$

A G-Brownian motion is firstly constructed on  $L_{ip}(\Omega_T)$ . For this purpose let  $(\xi_i)_{i \in \mathbb{N}}$  be a sequence of random variables on a sublinear expectation space  $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{E})$  such that  $\xi_i$  is G-normal distributed and  $\xi_{i+1}$  is independent of  $(\xi_1, \dots, \xi_i)$  for each integer  $i \geq 1$ .

Then a sublinear expectation on  $L_{ip}(\Omega_T)$  is constructed by the following procedure: For each  $X \in L_{ip}(\Omega_T)$  with  $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})$  for some  $\varphi \in C_{l,Lip}(\mathbb{R}^n)$ ,  $0 \leq t_0 < t_1 < \dots < t_n \leq T$ , set

$$E_G[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})] := \tilde{E}[\varphi(\sqrt{t_1 - t_0}\xi_1, \dots, \sqrt{t_n - t_{n-1}}\xi_n)].$$

The related conditional expectation of  $X \in L_{ip}(\Omega_T)$  as above under  $\Omega_{t_i}, i \in \mathbb{N}$ , is defined by

$$E_G[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}) | \Omega_{t_i}] := \psi(B_{t_1} - B_{t_0}, \dots, B_{t_i} - B_{t_{i-1}})$$

where  $\psi(x_1, \dots, x_i) := \tilde{E}[\varphi(x_1, \dots, x_i, \sqrt{t_{i+1} - t_i}\xi_{i+1}, \dots, \sqrt{t_n - t_{n-1}}\xi_n)]$ . One checks that  $E_G$  consistently defines a sublinear expectation on  $L_{ip}(\Omega_T)$  and the canonical process  $B$  represents a G-Brownian motion.

**Definition A.9** *The sublinear expectation  $E_G : L_{ip}(\Omega_T) \rightarrow \mathbb{R}$  defined through the above procedure is called a G-expectation. The corresponding canonical process  $(B_t)_{t \in [0, T]}$  on the sublinear expectation space  $(\Omega_T, L_{ip}(\Omega_T), E_G)$  is a G-Brownian motion.*

Let  $\|\xi\|_p := [E_G(|\xi|^p)]^{\frac{1}{p}}$  for  $\xi \in L_{ip}(\Omega_T), p \geq 1$ . Then for any  $t \in [0, T]$ ,  $E_G(\cdot | \Omega_t)$  can be continuously extended to  $L_G^p(\Omega_T)$  – the completion of  $L_{ip}(\Omega_T)$  under the norm  $\|\xi\|_p$ .

**Proposition A.10** *The conditional G-expectation  $E_G(\cdot | \Omega_t) : L_G^1(\Omega_T) \rightarrow L_G^1(\Omega_t)$  defined above has the following properties: For any  $t \in [0, T]$ ,  $X, Y \in L_G^1(\Omega_T)$  we have*

$$(i) \quad E_G(X | \Omega_t) \geq E_G(Y | \Omega_t) \text{ if } X \geq Y.$$

$$(ii) \quad E_G(\eta | \Omega_t) = \eta \text{ if } \eta \in L_G^1(\Omega_t).$$

- (iii)  $E_G(X|\Omega_t) - E_G(Y|\Omega_t) \leq E_G(X - Y|\Omega_t)$ .
- (iv)  $E_G(\eta X|\Omega_t) = \eta^+ E_G(X|\Omega_t) + \eta^- E_G(-X|\Omega_t)$  for each bounded  $\eta \in L_G^1(\Omega_t)$ .
- (v)  $E_G(E_G(X|\Omega_t)|\Omega_s) = E_G(X|\Omega_{t \wedge s})$ .
- (vi)  $E_G(X|\Omega_t) = E_G(X)$  for each  $L_G^1(\Omega_T^t)$ .

The following property is often very useful. Of course, it holds for any sublinear expectation if the related conditional expectation is defined reasonably.

**Proposition A.11** *Let  $X, Y \in L_G^1(\Omega_T)$  with  $E_G(Y|\Omega_t) = -E_G(-Y|\Omega_t)$  for some  $t \in [0, T]$ . Then we have*

$$E_G(X + Y|\Omega_t) = E_G(X|\Omega_t) + E_G(Y|\Omega_t).$$

*In particular, if  $E_G(Y|\Omega_t) = E_G(-Y|\Omega_t) = 0$  then we have  $E_G(X + Y|\Omega_t) = E_G(X|\Omega_t)$ .*

G-expectation and its corresponding G-Brownian motion is not based on a given classical probability measure. The next theorem establishes the ramification with probability theory. As a consequence we obtain a set of probability measures which represents the functional  $E_G$  in a subsequently announced sense. Although the measures belonging to the set are mutually singular this result is similar to the classical ambiguity setting when the probability measures inducing the ambiguity are absolutely continuous, see Chen and Epstein (2002), Delbaen (2002). References for the representation theorem for G-expectation are Denis, Hu, and Peng (2010) and Hu and Peng (2010).

Let  $\mathcal{F} = \mathcal{B}(\Omega_T)$  be the Borel  $\sigma$ -algebra and consider the probability space  $(\Omega_T, \mathcal{F}, P)$ . Let  $W = (W_t)$  be a classical Brownian motion in this space. The filtration generated by  $W$  is denoted by  $(\mathcal{F}_t)$  where  $\mathcal{F}_t := \sigma\{W_s | 0 \leq s \leq t\} \vee \mathcal{N}$  and  $\mathcal{N}$  denotes the collection of  $P$ -null subsets. For fixed  $t \geq 0$  we also denote  $\mathcal{F}_s^t := \sigma\{W_{t+u} - W_t | 0 \leq u \leq s\} \vee \mathcal{N}$ .

Let  $\Theta := [\underline{\sigma}, \bar{\sigma}]$  such that  $G(y) = \frac{1}{2} \sup_{\theta \in \Theta} y\theta^2$  and denote by  $\mathcal{A}_{t,T}^\Theta$  the collection of all  $\Theta$ -valued  $(\mathcal{F}_s^t)$ -adapted processes on  $[t, T]$ . For any  $\theta \in \mathcal{A}_{t,T}^\Theta$  we define

$$B_T^{t,\theta} := \int_t^T \theta_s dW_s.$$

Let  $P^\theta$  be the law of the process  $B_t^{0,\theta} = \int_0^t \theta_s dW_s, t \in [0, T]$ , i.e.,  $P^\theta = P \circ (B^{0,\theta})^{-1}$ . Define  $\mathcal{P}_1 := \{P^\theta | \theta \in \mathcal{A}_{0,T}^\Theta\}$  and (the weakly compact set)  $\mathcal{P} := \overline{\mathcal{P}_1}$  as the closure of  $\mathcal{P}_1$  under the topology of weak convergence.

Using these notations we can formulate the following result.

**Theorem A.12** *For any  $\varphi \in C_{l,Lip}(\mathbb{R}^n), n \in \mathbb{N}, 0 \leq t_1 \leq \dots \leq t_n \leq T$ , we have*

$$\begin{aligned} E_G[\varphi(B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})] &= \sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E^P[\varphi(B_{t_1}^{0,\theta}, \dots, B_{t_n}^{t_{n-1},\theta})] \\ &= \sup_{\theta \in \mathcal{A}_{0,T}^\Theta} E^{P^\theta}[\varphi(B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})] \\ &= \sup_{P^\theta \in \mathcal{P}} E^{P^\theta}[\varphi(B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})]. \end{aligned}$$

Furthermore,

$$E_G(X) = \sup_{P \in \mathcal{P}} E^P(X) \quad \forall X \in L_G^1(\Omega_T).$$

The last theorem can also be extended to the conditional G-expectation, see also Soner, Touzi, and Zhang (2010a). For  $X \in L_G^1(\Omega_T), t \in [0, T]$ , and  $Q \in \mathcal{P}$ ,

$$E_G(X|\mathcal{F}_t) = \operatorname{ess\,sup}_{Q' \in \mathcal{P}(t,Q)} E^{Q'}(X|\mathcal{F}_t) \quad Q - a.s.$$

where  $\mathcal{P}(t, Q) := \{Q' \in \mathcal{P} | Q' = Q \text{ on } \mathcal{F}_t\}$ .

As seen in the previous sections the following terminology is very useful within the framework of G-expectation.

**Definition A.13** *A set  $A \in \mathcal{F}$  is polar if  $P(A) = 0$  for all  $P \in \mathcal{P}$ . We say a property holds “quasi-surely” (q.s.) if it holds outside a polar set.*

Peng (2010) also gives a pathwise description of the space  $L_G^p(\Omega_T)$ . This is quite helpful to get a better understanding of the space. Before passing to the description we need the following definition.

**Definition A.14** *A mapping  $X : \Omega_T \rightarrow \mathbb{R}$  is said to be quasi-continuous (q.c.) if  $\forall \varepsilon > 0$  there exists an open set  $O$  with  $\sup_{P \in \mathcal{P}} P(O) < \varepsilon$  such that  $X|_{O^c}$  is continuous.*

*We say that  $X : \Omega_T \rightarrow \mathbb{R}$  has a quasi-continuous version if there exists a quasi-continuous function  $Y : \Omega_T \rightarrow \mathbb{R}$  with  $X = Y$  q.s.*

Peng (2010) showed that  $L_G^p(\Omega_T), p > 0$ , is equal to the closure of the continuous and bounded functions on  $\Omega_T$ ,  $C_b(\Omega_T)$ , with respect to the norm  $\|X\|_p := (\sup_{P \in \mathcal{P}} E^P[|X|^p])^{\frac{1}{p}}$ . Furthermore, the space  $L_G^p(\Omega_T), p > 0$ , is characterized by

$$L_G^p(\Omega_T) = \{X \in L^0(\Omega_T) : X \text{ has a q.c. version, } \lim_{n \rightarrow \infty} \sup_{P \in \mathcal{P}} E^P[|X|^p 1_{\{|X| > n\}}] = 0\} \quad (9)$$

where  $L^0(\Omega_T)$  denotes the space of all measurable real-valued functions on  $\Omega_T$ .

The mathematical framework provided enables the analysis of stochastic processes for several mutually singular probability measures simultaneously. Therefore, when not stated otherwise all equations are also to be understood in the sense of “quasi-sure”. This means that a “property” holds almost-surely for all conceivable scenarios.

## A.2 Stochastic calculus of Itô type with G–Brownian motion

We briefly present the basic notions on stochastic calculus like the construction of Itô’s integral with respect to G–Brownian motion.

For  $p \geq 1$ , let  $M_G^{p,0}(0, T)$  be the collection of all simple processes  $\eta$  of the following form: Let  $\{t_0, t_1, \dots, t_N\}, N \in \mathbb{N}$ , be a partition of  $[0, T]$ ,  $\xi_i \in L_G^p(\Omega_{t_i}) \forall i = 0, 1, \dots, N-1$ . Then for any  $t \in [0, T]$  the process  $\eta$  is defined by

$$\eta_t(\omega) := \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t). \quad (10)$$

For each  $\eta \in M_G^{p,0}(0, T)$  let  $\|\eta\|_{M_G^p} := \left(E_G \int_0^T |\eta_s|^p ds\right)^{\frac{1}{p}}$  and denote by  $M_G^p(0, T)$  the completion of  $M_G^{p,0}(0, T)$  under the norm  $\|\cdot\|_{M_G^p}$ .

**Definition A.15** For  $\eta \in M_G^{2,0}(0, T)$  with the presentation in (10) we define the integral mapping  $I : M_G^{2,0}(0, T) \rightarrow L_G^2(\Omega_T)$  by

$$I(\eta) = \int_0^T \eta(s) dB_s := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}).$$

Since  $I$  is continuous it can be continuously extended to  $M_G^2(0, T)$ . The integral has similar properties as in the classical Itô calculus case. For more details see Peng (2010).

The quadratic variation process of  $B$  is defined like in the classical case as the limit of the quadratic increments. The following identity holds

$$\langle B \rangle_t = B_t^2 - 2 \int_0^t B_s dB_s \quad \forall t \leq T.$$

The quadratic variation of  $B$  is a continuous, increasing process which is absolutely continuous with respect to  $dt$ .  $(\langle B \rangle_t)$  contains all the statistical uncertainty of  $B$ . It is a typical process with mean uncertainty. For  $s, t \geq 0$  we have  $\langle B \rangle_{s+t} - \langle B \rangle_s \sim \langle B \rangle_t$  and it is independent of  $\Omega_s$ . Furthermore, for any  $t \geq s \geq 0$

$$\begin{aligned} E_G[\langle B \rangle_t - \langle B \rangle_s | \Omega_s] &= \bar{\sigma}^2(t - s), \\ E_G[-(\langle B \rangle_t - \langle B \rangle_s) | \Omega_s] &= -\underline{\sigma}^2(t - s). \end{aligned}$$

We say that  $\langle B \rangle_t$  is  $N([\underline{\sigma}^2 t, \bar{\sigma}^2 t] \times \{0\})$ -distributed, i.e., for all  $\varphi \in C_{l,Lip}(\mathbb{R})$ ,

$$E_G[\varphi(\langle B \rangle_t)] = \sup_{\underline{\sigma}^2 \leq v \leq \bar{\sigma}^2} \varphi(vt).$$

The integral with respect to the quadratic variation of G-Brownian motion  $\int_0^t \eta_s d\langle B \rangle_s$  is defined in an obvious way. Firstly, for all  $\eta \in M_G^{1,0}(0, T)$  and again by a continuity argument for all  $\eta \in M_G^1(0, T)$ .

The following observation is important for the characterization of G-martingales. The Itô integral can also be defined for the following processes, see Song (2010b): Let  $H_G^0(0, T)$  be the collection of processes  $\eta$  having the following form: For a partition  $\{t_0, t_1, \dots, t_N\}$  of  $[0, T]$ ,  $N \in \mathbb{N}$ , and  $\xi_i \in L_{ip}(\Omega_{t_i}) \forall i = 0, 1, \dots, N-1$ , let  $\eta$  be given by

$$\eta_t(\omega) := \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t) \quad \forall t \leq T.$$

For  $p \geq 1$  and  $\eta \in H_G^0(0, T)$  let  $\|\eta\|_{H_G^p} := \left( E_G \left( \int_0^T |\eta_s|^2 ds \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}$  and denote by  $H_G^p(0, T)$  the completion of  $H_G^0(0, T)$  under this norm  $\|\cdot\|_{H_G^p}$ . In the case  $p = 2$  the spaces  $H_G^2(0, T)$  and  $M_G^2(0, T)$  coincide. As before we can construct Itô's integral  $I$  on  $H_G^0(0, T)$  and extend it to  $H_G^p(0, T)$  for any  $p \geq 1$  continuously, hence  $I : H_G^p(0, T) \rightarrow L_G^p(\Omega_T)$ .

### A.3 Characterization of G–martingales

**Definition A.16** A process  $M = (M_t)$  with values in  $L_G^1(\Omega_T)$  is called *G–martingale* if  $E_G(M_t|\mathcal{F}_s) = M_s$  for all  $s, t$  with  $s \leq t \leq T$ . If  $M$  and  $-M$  are both *G–martingales*  $M$  is called a *symmetric G–martingale*.

By means of the characterization of the conditional G–expectation we have that  $M$  is a G–martingale if and only if for all  $0 \leq s \leq t \leq T, P \in \mathcal{P}$ ,

$$M_s = \operatorname{ess\,sup}_{Q' \in \mathcal{P}(s, P)} E^{Q'}(M_t|\mathcal{F}_s) \quad P - a.s.$$

cf. Soner, Touzi, and Zhang (2010a). This identity declares that a G–martingale  $M$  can be seen as a multiple prior martingale which is a supermartingale for any  $P \in \mathcal{P}$  and a martingale for an optimal measure.

The next results give a characterization for G–martingales.

**Theorem A.17** Let  $x \in \mathbb{R}, z \in M_G^2(0, T)$  and  $\eta \in M_G^1(0, T)$ . Then the process

$$M_t := x + \int_0^t z_s dB_s + \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds, \quad t \leq T,$$

is a *G–martingale*.

In particular, the nonsymmetric part  $-K_t := \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s) ds$ ,  $t \in [0, T]$ , is a G–martingale which is quite surprising compared to classical probability theory since  $(-K_t)$  is continuous, non–increasing with quadratic variation equal to zero.

**Remark A.18**  $M$  is a symmetric *G–martingale* if and only if  $K \equiv 0$ , see also Song (2010b).

**Theorem A.19 (Martingale representation)** (Song (2010b)) Let  $\beta \geq 1$  and  $\xi \in L_G^\beta(\Omega_T)$ . Then the *G–martingale*  $X$  with  $X_t := E_G(\xi|\mathcal{F}_t), t \in [0, T]$ , has the following unique representation

$$X_t = X_0 + \int_0^t z_s dB_s - K_t$$

where  $K$  is a continuous, increasing process with  $K_0 = 0, K_T \in L_G^\alpha(\Omega_T), z \in H_G^\alpha(0, T), \forall \alpha \in [1, \beta)$ , and  $-K$  a *G–martingale*.

If  $\beta = 2$  and  $\xi$  bounded from above we get that  $z \in M_G^2(0, T)$  and  $K_T \in L_G^2(\Omega_T)$ , see Song (2010a).



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